A PROBLEM IN RELATIVISTIC NAVIGATION: THE THREE-DIMENSIONAL ROCKET EQUATION

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Several causes alter a thrust vector with respect to the nominal direction of flight. A three-dimensional equation of motion of a starship becomes then necessary for a more realistic analysis of a mission. In this paper a generalisation for a pure-rocket ship is made and the fundamental problem of navigation in interstellar flight is quantitatively dealt with.

Although preliminary, a scheme is suggested in order to calculate the state vector in a galactic frame starting from the output of on-board clocks and accelerometers. A technique to update the ship's clock is also discussed.

1. INTRODUCTION

IT IS WELL-KNOWN that the leading problem of interstellar flight, i.e. the propulsion system, is far from being solved. New concepts and technological breakthroughs are claimed. However, at the "state-of-art of theoretical studies" we can cite two propulsion concepts which may be applicable in the near term: the Project Daedalus concept of the BIS and the fission-drive concept [1-3]. Performance and limitations of both have been found out. Therefore, it is not too premature to begin viewing a propulsion system, also from a Navigation performance point of view. The author has concerned himself exclusively with the latter propulsion concept. However, the concepts developed in this paper can be applied to any internally-powered star-vehicle, especially if a continuous-thrust system over a certain time interval represents its primary propulsion.

The equation of relativistic motion of a rocket has been formulated in such a way to take several important aspects into account [1]; but even that generalised form lies on one-dimensional considerations. In particular, having dealt with the motion of the centre-of-mass, there was the restriction that the X-axis of the instantaneous rocket-frame, which is always parallel to the starship velocity, "slips" on the X-axis of the heliocentric frame of reference. As a consequence, the vector exhaust speed is antiparallel to the velocity of an accelerating ship over the whole thrusting time. It's obvious that, because of unavoidable errors at launch and during power phases, the starship in reality points off its nominal rectilinear trajectory and therefore one or more corrections are necessary; the starship will appear to undergo a three-dimensional thrust in a Sun frame. In addition, even if these errors were eliminated or greatly reduced, it is hard to manage to keep the angle between jet and nominal trajectory strictly at 180° (or 0° for a nominal deaccelerating ship). For instance, referring us to one of the missions to Barnard's Star in Ref. 2 (flight time = 54 years, fission-engine propulsion time = 3.75 years, mean exhaust speed = 10,900 km/s) elementary considerations tell us that a hypothetical constant misalignment of $\theta$ between the axis of the magnetic nozzle and the nominal direction of flight would cause a delivery error equal to 110 AU (Solar System "radius" $\approx 9$ AU). Notice that the main sources of error in some studies of terminal navigation in interplanetary Solar-Electric propulsion missions consist of long term thrust noise and thrust vector direction noise. Something analogous may well occur when employing ejection of fission fragments directionally by a magnetic field. Even though the above cited flight is a very fast fly-by, an unsteered ship should carry a Palomar-like telescope on-board in order to be able to detect particulars of interest of the planets circling Barnard's Star. An equivalent mass or less could be employed for the success of the mission (especially if a microminiaturisation were required as discussed in Ref. 2). Moreover, if correction thrusts are necessary, e.g. non-parallel to the ship's speed and lasting a long time, a further error can easily be made when aligning the magnetic nozzle (if any; not necessarily of the primary thruster) along the appropriate direction in the ship-frame at the time considered. An undesired component of the velocity increment needs to be evaluated and successively eliminated. Thus, from the above considerations the first problem already arises: whenever the ship's velocity is misaligned with respect to the X-axis of the galactic frame of reference (which generally represents the nominal direction of flight) and/or to the exhaust speed, how does one write the rocket equation? In the next section we begin by coping with this important question, extending the Einstein-Lorentz transformation to the case where the ship's speed is randomly oriented in the heliocentric frame. The reader can verify that several considerations will be of general character.

2. GENERAL EINSTEIN-LORENTZ TRANSFORMATIONS

In order to achieve this goal we make the following rotations of the instantaneous-at-rest rocket frame $(S' \equiv O', X', Y', Z')$ (see Fig. 1), usually defined as [1]:

(a) $S'$ rotates through an angle $\phi$ about its Z-axis in order to overlap its X-axis on the projection of $Y$ (the starship's velocity at the time considered in the heliocentric frame) on the $X'$-$Y'$-plane. Thus, we have a second (intermediate) frame $S'' \equiv [X', Y', Z'' \equiv Z']$.

(b) $S''$ rotates through an angle $\Theta$ about $Y''$ such that $X''$ coincides with $Y$, Therefore the third frame is $S''' \equiv [O', X''' \equiv Y', Y''' \equiv Y', Z''']$. Notice that this rotation procedure implies that if $\phi$ is counterclockwise, $\Theta$ is clockwise and vice versa.

The space coordinate transformation is written as

$$
\begin{bmatrix}
    x'''
    y'''
    z'''
\end{bmatrix} = \Omega (\phi, \Theta) \begin{bmatrix}
    x'
    y'
    z'
\end{bmatrix} \ast
$$

where $\Omega$ is a 3 x 3 matrix, $\ast$ denoting the transpose of a
and to keep its history in order to determine the ship’s state-vector in the heliocentric frame. The determination of \( V(t) \) is actually a part of a more complex procedure, the result of which is represented by the “estimated navigation state vector” referred to a Sun frame \([4, 5]\). We’ll return to such a question later.

3. KINEMATICAL CONSEQUENCES AND ROCKET EQUATION

Finally we drop the restriction \( V = \text{const} \) in the formulæ we intend to apply to an accelerating/decelerating starship. By means of a well-known procedure of Special Relativity it is possible to carry out the law of velocity composition and the expression of the acceleration in the Sun frame as follows:

\[
\begin{align*}
\frac{d\mathbf{u}}{dt} &= \gamma \left( 1 + \mathbf{V} \cdot \mathbf{a}' \right) \frac{d\mathbf{a}'}{dt} \\
\mathbf{a} &= \Xi \left( \mathbf{a}' - \gamma \mathbf{V} (\mathbf{V} \cdot \mathbf{a}') \right)/\left(1 + \mathbf{V} \cdot \mathbf{a}' \right)^2 \gamma^2
\end{align*}
\]

where \( \mathbf{u}' \) and \( a' \) are velocity and acceleration in \( S' \) and \( S \) respectively. Let us apply the laws expressed by Eqs. (9) to the powered motion of a starship. Because \( u' = 0 \) we have:

\[
\begin{align*}
\frac{d\mathbf{u}}{dt} &= \gamma \mathbf{V} \\
\mathbf{a} &= \Xi \left( \mathbf{a}' - \gamma \mathbf{V} \mathbf{a}' \right)/\gamma^2
\end{align*}
\]

The first two relations are obvious. In contrast, the third expression assumes a non-simple link-up between the Sun-frame starship’s acceleration, i.e. \( a = \mathbf{dv}/dt \), and the inertial acceleration \( a'_b \) sensed by a system of accelerometers on-board. This includes the contribution from any kind of acceleration but the gravitational one*, if any (this doesn’t disagree with the Principle of Equivalence, of course). A useful result from the above acceleration equation is given by

\[
\mathbf{V} \cdot \mathbf{a} = \mathbf{V} \cdot \left[ \gamma^2 a'_b \right].
\]

Usually in Special Relativity acceleration is not given much importance; however, in this particular case we can uniquely link a quantity measurable on-board with the “thrust-field” which the starship is experiencing.

If \( M \) denotes the rest-mass of the ship, defining effective thrust as the time derivative of its momentum, a little matrix algebra enables us to obtain the law of force transformation as follows:

\[
F = \gamma^2 \Xi F_b
\]

where \( F_b = M a'_b \). We see that in the general case the force measured as an inertial on-board quantity has neither the same magnitude nor the same direction with respect to the corresponding one in the Sun frame. This is a result quite different from that of the one-dimensional case \([4]\). In addition, comparing Eq. (10c) with Eq. (12), \( F \) does not result in pointing in the same direction as the acceleration \( a \) does.

* (See Appendix C for a further discussion).
† The cited number of dimensions represents that of non-zero components of \( V \) in \( S \).

The general equation of relativistic motion becomes

\[
F = \left[ \gamma^2 (V \cdot a) V + \gamma V \frac{dV}{dt} \right] M
\]

whereas the power associated with \( F \) is

\[
V \cdot F = \gamma^2 (V \cdot a) = (V \cdot a'_b) M
\]

or, equivalently,

\[
V/V \cdot F = V/V \cdot F_b
\]

Therefore, despite the complexity of Eq. (12), the components of the inertial force and its corresponding effective thrust (Sun frame) along the instantaneous direction of the starship’s velocity are equal to each other. This represents the generalisation of the one-dimensional interrelationship between the two measurements of thrust.

With all this in mind, we are now ready to contend with the question of writing the three-dimensional equation of motion of an internally-powered vehicle. Keeping the symbols of Ref. 1 (also see Appendix B), we have the laws of conservation of energy and impulse respectively as follows:

\[
\begin{align*}
dE_R + dE_P + dE_I &= 0 \\
dE_P + dP_d + dP_i &= 0
\end{align*}
\]

where ‘R’, ‘P’ and ‘I’ stand for rocket, propellant and loss of mass respectively \([1]\). Proceeding similarly to Ref. 1 and taking into account both the transformation law Eq. (9b) and Eq. (A-9), we obtain

\[
V \gamma^2 \frac{dV}{dt} = \eta_m G_{Vj} \gamma_{Vj} \left( V \cdot V \right) dM/M
\]

where \( V_j \) is the jet speed (in \( S \)), \( \eta_m \) the propellant efficiency and \( G_{Vj} \) the interaction function defined in Ref. 1. Eq. (14) gives the increment of the magnitude of \( V \), but its integration requires \( V/V \). A second equation must be derived. This is performed starting from Eq. (13b). Omitting the presentation of intermediate calculus, the final result is the following:

\[
\begin{align*}
\frac{dV}{dt} &= \gamma \eta_m G_{Vj} \left( V \cdot V \right) dV + \eta_m G_{Vj} \gamma_{Vj} \gamma^2 \Xi Y_{Vj} dM/M \\
\gamma \frac{dV}{dt} &= -\eta_m G_{Vj} \left( V \cdot V \right) \gamma_{Vj} \Xi Y_{Vj} \eta_m dM/M
\end{align*}
\]

Let us note how \( dV \) depends upon \( dV \). Eqs. (14) and (15) translate the conservation of energy and momentum respectively; only in the one-dimensional case do these equations coincide with each other. Because the factor \( V \gamma^2 \frac{dV}{dt} \) is present in the left-hand side of Eq. (14), as in the right-hand side of Eq. (15), substitution of the former for the latter gives us the following equation:

\[
\begin{align*}
\frac{dV}{dt} &= -\eta_m G_{Vj} \left( V \cdot V \right) \eta_{Vj} \gamma^2 \Xi Y_{Vj} dM/M \\
\gamma \frac{dV}{dt} &= -\eta_m G_{Vj} \gamma_{Vj} \eta_{Vj} dM/M
\end{align*}
\]

The above equation is the rocket vector equation we are looking for. It contains physical quantities such as \( V_j \), \( M \) and \( G_{Vj} \) which are referred to the same frame – the starship frame – the orientation of which is continuously or periodically corrected in order to maintain its initial attitude chosen parallel to the orientation of the heliocentric frame to which the starship’s velocity \( V \) is referred. In the next section we’ll modify Eq. (16) in order for it to be more directly exploited by a Navigation loop.

Note that if a perfect parallelism between \( V_j \) and \( V \) were possible, Eqs. (14) and (15) would result respectively in:

\[
\begin{align*}
\gamma \frac{dV}{dt} &= \eta_m G_{Vj} \gamma_{Vj} V_j dM/M \\
\gamma \frac{dV}{dt} &= \eta_m G_{Vj} \gamma_{Vj} dM/M
\end{align*}
\]

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vector/matrix. Before going on, let us note that S' is parallel
to the heliocentric frame S_h (and \(t_h = t' = 0\) only when
\(O_h \equiv O_S\)). This is not merely a mathematical choice; during
the flight an on-board gyro-computer system is provided to
maintain the rocket-frame parallel to its initial orientation
and the phenomena observed by and/or detected in the ship
are quantitatively referred to such attitude.

A similar procedure of rotation can be applied to S_h, to
give a new frame S_{h'} which is parallel to S''. These considera-
tions allow us to set down

\[
\begin{pmatrix}
x' \ y' \ z'
\end{pmatrix}^* = \Omega(\varphi, \Theta) \begin{pmatrix}
x_h \ y_h \ z_h
\end{pmatrix}
\]

Assuming— for now— \(V = \text{const.}\), we write the familiar E-L
transformations

\[
\begin{align*}
x' &= \gamma_V (x'' + Vt'') \\
y' &= y'' \\
z' &= z'' \\
t' &= \gamma_V (t''' + Vx'')
\end{align*}
\]

where the velocity of light, c, is defined as unity, and \(\gamma_V = (1 - V^2)^{-1/2}\). Taking into account that \(t''' = t'\) and \(t_h = t_h;\) we combine Eqs. (1), (2), and (3) to give

\[
\begin{pmatrix}
x_h \\
y_h \\
z_h
\end{pmatrix} = \Omega(-\Theta, \varphi) A \begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}
\]

\[
t_h = \gamma_V (t' + Vt')
\]

where we have set

\[
A = \begin{pmatrix}
\gamma_V & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and used the equality

\(\Omega^{-1}(\varphi, \Theta) = \Omega(-\Theta, \varphi)\).

In the fourth of the Eqs. (4) \(t'\) is the vector radius point-
ing from \(O'\) toward the space-point considered in \(S'\). (For
simplicity, from now on let us suppress the subscript "h").

From the theory of the matrices of rotation we have the following expression of \(\Omega:\)

\[
\Omega(\varphi, \Theta) = \begin{pmatrix}
\cos \Theta \cos \varphi & \sin \Theta \sin \varphi & \cos \Theta \\
-sin \varphi & \cos \Theta \sin \varphi & 0 \\
-cos \Theta \sin \Theta & -sin \varphi \sin \Theta & \cos \Theta
\end{pmatrix}
\]

A clockwise rotation means a negative angle to be ins€rted
into Eq. (5). (Remember that the first row gives the direc-
tion cosines of \(V\) with respect to \(S'\)). Similarly we have \(\Omega^{-1}(\varphi, \Theta)\). From Fig. 1 we have, by construction

\[
\begin{align*}
\sin \Theta &= V_z/V \\
\cos \varphi &= V_x/(V_x^2 + V_y^2)^{1/2}
\end{align*}
\]

Now we can proceed to calculate the terms in Eq. (4). The result is the following:

\[
\Omega(-\Theta, \varphi) = \begin{pmatrix}
\gamma_V & 0 & V_x \\
0 & \gamma_V & V_y \\
0 & 0 & \gamma_V
\end{pmatrix}
\]

\[
\Omega(-\Theta, \varphi) A \Omega(\varphi, \Theta) = \Xi (V_x, V_y, V_z)
\]

Because of both the importance and the complexity of the
symmetric matrix \(\Xi\), we furnish its explicit terms and
specify some important properties in Appendix A, while
from now on we retain only its symbol. We name it the
Space-Transformation matrix. By means of the relations in
Eqs. (7) we can re-arrange the transformations in Eq. (4) in
the following manner:

\[
\begin{pmatrix}
x \\
y \\
z \\
t
\end{pmatrix} = \begin{pmatrix}
V_x & \gamma_V \\
V_y & \gamma_V \\
V_z & \gamma_V \\
0 & 1
\end{pmatrix} \begin{pmatrix}
x' \\
y' \\
z' \\
t'
\end{pmatrix}
\]

Equation (8) represents a generalised coordinates' trans-
formation with \(V = \text{const.}\). Notice that the frame \(S'\) can
always be obtained by a pure translation of \(S\) independently
of the history of \(V\). More concisely, we can write Eq. (8) as
follows

\[
P = \Phi P'
\]

where \(P\) denotes a four-vector expressing the "kinematic
state" of the event: \(P^* = (x, y, z, t)\). However, it may be use-
ful to separate space from time transformations. In fact,
they can be resolved in

\[
\begin{align*}
\tau &= \Xi t' + \gamma_V V t' \\
t &= \gamma_V (t' + Vt')
\end{align*}
\]

which are easier to handle subsequently. If we recall the
property of \(\Xi\) expressed by Eq. (A-3), we can write the
space-equations as

\[
\tau = \Xi (t' + V t'),
\]

from which one sees that the role of the Lorentz factor \(\gamma_V\)
in the one-dimensional case is replaced by the matrix \(\Xi\) in
the three-dimensional one. The time transformation appears
as an obvious extension. One of the purposes of the com-
puter system of the ship is to both update \(\Xi\) in its memory,
In this ideal case the complex effect of $\Xi$ vanishes and $d\mathbf{V}$ is parallel to $-\mathbf{V}_j$, but only here.

Returning to the general case, we first differentiate both sides of Eq. (16) with respect to $t$; then we make the scalar product with $\mathbf{V}$ and substitute them (i.e. $a$ and $a\mathbf{V}$) in Eq. (12a). This gives

$$T = \gamma_m \mathbf{G}_{\mathbf{V}_j} \gamma_j \mathbf{M} \gamma_j^2 \Xi \mathbf{V}_j$$

where the mark (') means a differentiation with respect to the ship's proper time. At this point applying the operator $\gamma_j \Xi^{-1}$ on both sides of Eq. (17) results in

$$\mathbf{I}_S = \gamma_m \mathbf{G}_{\mathbf{V}_j} \gamma_j \mathbf{M} \gamma_j \mathbf{V}_j,$$  

where $\mathbf{I}_S$ denotes the "thrust" as measured on-board*. Remembering that $M = G^2 M_p$ (see Ref. 1), we can write it alternatively as

$$\mathbf{I}_S = -\gamma_m \mathbf{G}_{\mathbf{V}_j} \gamma_j \mathbf{M} \gamma_j \mathbf{V}_j,$$  

$M_p$ being the propellant flow-rate. From Eq. (17) we are induced to define a "vector effective jet speed" as follows:

$$\mathbf{V}_{\text{eff}} = \gamma_m \mathbf{G}_{\mathbf{V}_j} \gamma_j \mathbf{V}_j \gamma_j^2 \Xi \mathbf{V}_j$$

by means of which the expression of the thrust in a Sun frame re-assumes its familiar form again. In general, $\mathbf{V}_{\text{eff}}$ is not parallel to $\mathbf{V}_j$ unless $\mathbf{V}_j$ is parallel to $\mathbf{V}$. The rocket equation becomes

$$d\mathbf{V} = -[(\mathbf{V}_{\text{eff}} \cdot \mathbf{V}_j) \mathbf{V} - \mathbf{V}_{\text{eff}}] dM/M.$$  

Thus, the increment of the starship's velocity is actualised along the direction of the vector coefficient in Eq. (16a).

Leaving to the reader the manifold considerations about rocket energetics (in the wake of those done in Ref. 1), because our current goal lies in the framework of Navigation, we limit ourselves to noting that the time rate of the rocket energy is still given by

$$-\frac{dE}{dt} + \mathbf{T} \cdot \mathbf{V} = -\mathbf{M}.$$  

Or, since $\mathbf{M}$ is always negative, we find again that the time rate of the infinitesimal work done by $\mathbf{T}$ is constantly greater than the temporal variation of the total energy of a rocket-ship because of the ejection of matter.

4. SHIP-STATE FROM ON-BOARD MEASUREMENTS

In this section we arrange the general results obtained previously in order to find a procedure enabling us to calculate the indicated state vector of the starship in the Sun frame, starting from and essentially exploiting the output of clocks and accelerometers. We are explicitly dealing with no other manoeuvre if improved during the long time of travel of the starship. Therefore it becomes possible by integration to retrace the velocity and vector position of the ship without knowing the propulsion ratio history. Besides, integrating the fourth E-L transformation, the galactic time is obtained.

Since the unit vector $\mathbf{V}_j/\mathbf{V}_j$ is the opposite of $\mathbf{V}_{\text{eff}}$, now $\mathbf{V}_{\text{eff}}$ can be found from Eq. (22), (17a) and by means of $\mathbf{V}$. In so doing, we have

$$\mathbf{V}_{\text{eff}} = -\mathbf{V}_j \gamma_j^2 \Xi (\mathbf{a}_a/\mathbf{a}_a).$$

* One pictures a force antiparallel to $\mathbf{V}_j$ responsible for the accelerated motion. An analogous consideration is done with respect to $\mathbf{a}_a$.

† (See Appendix C for a further discussion).
Let us point out that the whole procedure here described is — apart from the hypotheses regarding the model of the ejection of matter — based on the knowledge of a sequential set of measurements of both on-board acceleration and time.

Summarising, the following augmented state vector can be brought forth:

\[
\Delta = \left[ \begin{array}{c} t \, v \, (M_{\gamma y}) \, (M_{\nu \nu}) \end{array} \right]
\]

The vector \( \Delta \) can be further augmented with other quantities of importance in Guidance and Navigation processes. We limit ourselves to a discussion of its current size. The summarising scheme of Fig. 2 shows the utilisation of "inertially-sensed data" in order to compute the state vector of the ship. The very complex and quite important block, indicating a process for improving \( \Delta \) by means of periodic on-board measurements of external objects, and the Guidance block have been added for the sake of completeness. Some general indications about them can be found in Ref. 5. Finally let us notice that in evaluating the state vector one should account for the effect of delay stemming from a finite computation time, especially when Guidance must be performed according to certain prefixed figures of merit, particularly during powered approaching phases. Such a delicate topic will certainly represent a key-subject in a future development of both criteria and policies in Interstellar Navigation, Guidance and Control.

In the next section we will treat the utilisation of on-board data from a slightly-different point of view. To avoid unnecessary complexity and hence give a numerical example, we restrict our considerations within the one-dimensional case.

5. WORLD-LINE OF A STARSHIP UNDER CONSTANT THRUST

The relativistic effects on a starship can be synthesised and visualised in a different manner, namely, through an analysis of its world-line. Since one may have an interest in a staged vehicle, we begin by considering a generic stage, the thrust of which is assumed constant over its firing time. This is a case mathematically different from that of a constant acceleration (which is very difficult to realise) solvable in closed-form [5]. For simplicity, we first deal with a stage jettisoning no amount of non-reusable matter in the time interval considered. Since we limit ourselves to a two-dimensional analysis, our effort consists of deriving a relationship between galactic time and distance travelled. Although such an equation admits no closed-solution, the importance of a world-line lies on the fact that it contains quantities measurable on-board which, passing through such equation, give information about the starship in a galactic frame.

If once again \( a_s \) and \( r \) denote the acceleration measured by a system of accelerometers and the rocket-clock readout respectively, then we have

\[
a_s = F/M(x) = \eta M \gamma y V_j M_p/(1 - G^1 M_p r) .
\]

The Lorentz factor can be expressed as function of \( r, M_p, V_j \) and \( V_{je} = V_{je} \) in the one-dimension case as follows:

\[
\gamma(V(r)) = (q(1 + sr)^2 V_{je} + q^2(1 + sr)^2 V_{je})/2
\]

where we have set

\[
q = (1 + V_{in})/(1 - V_{in})^{1/2}, \; s = -G^1 M_p .
\]

Similarly, from

\[
t = \int_0^r \gamma(V(v)) \, dv
\]

and using Eq. (27), we obtain a closed-form for galactic time as follows:

\[
t = 1/2s \left[ (1 + V_{in})/(1 - V_{in})^{1/2} \right]
\]

which we write as

\[
t = \Lambda(x, V_{in}, a_s, V_{je}) = \Lambda(r).
\]

Because of its particular symmetry Eq. (30) is even valid for a decelerating stage provided that \( q \) is referred to its final speed. We clearly see that, whereas it is possible to determine \( t \) analytically from \( r \), the reversal can be accomplished only iteratively; let us denote this solution as \( \Lambda^{-1}(t) \).

In Ref. 1 we derived the following constant of motion

\[
L V_j \gamma_{yj}/(\gamma_{yj} - 1) = G^1 t/\eta M(\gamma_{yj} - 1) - (\gamma_{in} - \gamma_{in} R^1)/\eta \eta W_{sp}
\]

where \( W_{sp} = W/M_{in} \) and, as usual, \( R^1 = 1 + sr \). \( L \) denotes the distance travelled by that stage from its blast-off to time \( t \). Setting

\[
\gamma_{in} - \gamma_{in} R^1 = \Psi(t, a_s, V_{je}), \; R^1 = B(\Lambda^{-1}(t), a_s, V_{je})
\]

and re-arranging Eq. (31) by the aid of both Eqs. (26) and (27), we finally obtain

\[
L = t/V_{je} - \Psi(\Lambda^{-1}(t), a_s, V_{je})/a_s B(\Lambda^{-1}(t), a_s, V_{je})
\]

(always in interstellar units) where, we repeat, \( \Lambda^{-1}(t) \) denotes the numerical solution of the exponential equation

\[
t = \Lambda(r) = 0
\]

with respect to \( r \), having given \( t \). (Note that, when utilised on-board, Eq. (32) does not claim to solve Eq. (33) in as much as its solution is evidently known. The converse takes place in determining the nominal profile \( L(t) \). Eq. (32) is the two-dimension world-line of a starship under constant thrust with no change of proper mass. When a discontinuity
coming from some discrete jettisoning policy occurs, the interaction function in the last environment must be taken into account and it suffices if the number of pseudo-stages (a pseudo-stage has been defined in Ref. I as the physical stage operating between two successive jettisonings) is high enough with respect to the desired accuracy in evaluating L compatible with the inaccuracies of the other measured variables such as r and aₚ [1]. If both that number is low and the vehicle is multi-stage-arranged, Eq. (32) has to be applied pseudo-stage after pseudo-stage and stage by stage, resetting the origins of both L and t; varying mass is accounted for by the corresponding sharp variation of aₚ.

For an example, consider a six-stage spaceship flying toward a hypothetical target located at about 2 ly away from the Sun. Assuming an initial acceleration equal to .01 g for each stage, an effective jet speed equal to 9,000 km/s and an overall propulsion time lasting 6.8 yr. (three accelerating stages plus as many decelerating), the related trajectory in a two-dimension space-time diagram is displayed in Fig. 3. This world-line meets the top horizontal line representing the world-line of the target at a galactic time of about 33 yr.

Both the powered flight phases have been obtained from Eq. (32) accounting for a multi-jettison policy.

In the previous section we saw that fitting the sequence (aₚ, r) Vₚ is easy computed. Inserting these three values into Eq. (29), the galactic time t is obtained on-board. Let us note that actually such a value corresponds to the read-out τ. Is this time (we denote it τₛ) really the galactic time measurable in a Sun frame? It is not impossible that the clock system or its processing system has been failing in the last environment; the ensuing statistical analysis can tell us how much the trajectory of the ship is off-nominal. For such an occurrence there exist other methods of analysis to detect any mismatching, and afterwards produce a correction command.

It is clear that only the case (ii) is meaningful in inferring incorrect working of the clock system. In addition, observation of stars can enable the computer to update the on-board time value τₛ.

6. CONCLUSIONS

A concept of propulsion concerning an interstellar drive could be wholly or partially phased out if it is intrinsically inappropriate when realising the Navigation and Guidance accuracy a certain flight needs. Several causes such as a thrust vector noise, errors at launch and corrections to steer a starship make its trajectory three-dimensional, and this is also the case in relativistic environments. In this paper we have been concerned with two questions: first, to write the equation of motion of a pure rocket in the case of the ship velocity randomly oriented in a Sun frame; second, to analyse some aspects of the determination of the state vector. Having discarded problems inherent to the control of the orientation of the ship, we have seen that on-board measurements of both time and acceleration and the updated knowledge of the space-transformation matrix enable the computer of the ship to perform an estimation of an augmented state vector comprising the total ship's energy and the thrust in a Sun frame. A different analysis of relativistic effects on the motion of the vehicle has been
also accomplished in the one-dimensional case to illustrate the use of the world-line of a starship under constant thrust. The whole procedure followed here is independent of any particular concept of propulsion or specific device of accelerometer and clock.

APPENDIX A: SPACE TRANSFORMATION MATRIX

In this appendix we give the terms of $\Xi$ and point out some important properties of it. $\Xi$ is a $3 \times 3$ matrix as follows:

\[
\begin{align*}
\Xi_{11} &= \gamma_v \left( \frac{v_x^2}{V^2} + \left( \frac{v_y^2 + v_z^2}{V^2} \right) \left( \frac{V_x^2 + V_y^2}{V^2} \right) \right) \\
\Xi_{12} &= (\gamma_v - 1) V_x V_y / V^2 \\
\Xi_{13} &= (\gamma_v - 1) V_x V_z / V^2 \\
\Xi_{21} &= \Xi_{12} \\
\Xi_{22} &= \gamma_v \left( \frac{v_x^2}{V^2} + \left( \frac{v_y^2 + v_z^2}{V^2} \right) \left( \frac{V_x^2 + V_y^2}{V^2} \right) \right) \\
\Xi_{23} &= (\gamma_v - 1) V_y V_z / V^2 \\
\Xi_{31} &= \Xi_{13} \\
\Xi_{32} &= \Xi_{23} \\
\Xi_{33} &= (\gamma_v V_x^2 + V_y^2 + V_z^2) / V^2
\end{align*}
\]

(A-1)

The first properties of $\Xi$ can be summarised as follows:

\[
\det \Xi = \gamma_v, \quad \Xi = \Xi^\circ
\]

(A-2)

Let us apply the operator $\Xi$ to $V$ and any three-vector $W$. We find the following useful results:

\[
\begin{align*}
\Xi V &= \gamma_v V \Xi & (V' = \gamma_v V) \\
\Xi W &= (\gamma_v - 1) (W \cdot V) V / V^2 + W \\
(\Xi V)^\circ &= V^\circ \Xi \\
\gamma_v^2 \Xi W &= 1_3 W = W \text{ only if } W \parallel V
\end{align*}
\]

(A-3)

(A-4)

(A-4a)

(A-5)

(A-6)

(A-7)

(A-8)

The operator $\Xi$ acts on a three-component vector $W$ in such a way as to first rotate its frame through the angles $\phi$ and $\Theta$ (see Fig. 1), amplify its current $X$-coordinate by $\gamma_v$ and finally restore the frame itself through the rotation $-\Theta$ followed by $-\phi$. The overall effect is then to increase the magnitude of $W$, simultaneously reducing or augmenting its primitive angle $W/\gamma_v$ if it is acute or obtuse respectively. If we substitute $L'$ — a vector in the ship's frame — for $W$ and denote the corresponding vector in a Sun frame by $L$, its Lorentz factor is expressible as:

\[
\gamma_L = \gamma_v \left( 1 + L' \cdot V \right) \left( 1 - L'^\circ \Xi^\circ - \gamma_v^{1/2} \right) = \gamma_v \left( 1 + L' \cdot V \right) \gamma_L'
\]

(A-9)

that is, it behaves as in the one-dimensional case. In Eq. (A-9) we used the law of velocity composition applied to $L'$ and $V$. The above decomposition of $\gamma_L$ depends upon the scalar product $L' \cdot V$ but doesn't depend on the particular sequence of rotation to bring $S'$ to coincide its $X$-axis with $V$. The term in brackets in Eq. (A-9) is very important in order to write down the rocket equation in as concise a way as possible.

APPENDIX B: LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>acceleration in a Sun frame</td>
</tr>
<tr>
<td>$\dot{a}_s$</td>
<td>acceleration as evaluated from the output of accelerometers</td>
</tr>
<tr>
<td>$F$</td>
<td>force in a Sun frame</td>
</tr>
<tr>
<td>$G$</td>
<td>interaction function</td>
</tr>
<tr>
<td>$M$</td>
<td>rest-mass of the starship</td>
</tr>
<tr>
<td>$M_p$</td>
<td>propellant mass ratio</td>
</tr>
<tr>
<td>$\tau$</td>
<td>galactic time</td>
</tr>
<tr>
<td>$V$</td>
<td>starship velocity</td>
</tr>
<tr>
<td>$V_j$</td>
<td>exhaust speed</td>
</tr>
<tr>
<td>$W$</td>
<td>propulsion power</td>
</tr>
<tr>
<td>$I$</td>
<td>thrust</td>
</tr>
<tr>
<td>$\eta_m$</td>
<td>propellant efficiency</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Lorentz factor</td>
</tr>
<tr>
<td>$\tau$</td>
<td>rocket-clock readout</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>rotation matrix</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>space-transformation matrix</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>augmented state vector</td>
</tr>
</tbody>
</table>

Subscripts

- $v$ magnitude of vehicular speed
- $v_j$ magnitude of jet speed
- eff effective jet
- $s$, $b$ on-board
- $h$ heliocentric

APPENDIX C

In this appendix we discuss Eqs. (10c) and (23). One can easily verify by aid of Eqs. (10a) and (A-3) that Eq. (23) is exactly Eq. (10c) only if $\dot{a}_s = a_s$. This means two things:

(a) Eq. (10c) is more general than Eq. (23) or, equivalently, the rocket equation written explicitly in terms of both on-board-sensed acceleration and time is a particular case of the acceleration transformation law;

(b) an overall acceleration or the vehicle caused by a mixing of internal and non-internal forces can be distinguished from an internally-yielded force by the fact that the latter obeys an equation such as Eq. (21b) where the coefficient of the (proper) time is independent of any non-internal contribution. It's not difficult to prove that in the case of variable thrust, Eq. (21b) generalises to:

\[
\frac{1}{a_s} = \frac{1}{a_s^O} \frac{T_s}{T_s^{SO}} \cdot \frac{<T_s>}{T_s} \frac{\gamma}{V_j e}
\]

(C-1)

where $<T_s>$ denotes the mean value of $T_s$ over the $[0, \tau]$ interval.
When the starship enters a space zone where the on-board measurements (both direct and indirect) are not compatible with an equation such as Eq. (C-1), an external field (active or passive) is then inferred. Gravitation can be separated from the remaining field only computationally, once the position of the ship is known with respect to all attracting bodies. Other fields can be tested by the computer provided that some models of the expected fields have been built into its memory.

Therefore, a supplementary computational open loop must be considered in a Navigation algorithm in order to appropriately determine the space-transformation matrix.

REFERENCES
2. G. Vulpiani, 'Direct Fission Propulsion: Improvement of a Series-Staged Starship from Impulsive Jetting Policy', to be published in JBJS.

U.S. SYMPOSIUM DISCUSSSES SPACE COLONIZATION, PROSPECTS FOR LIFE IN THE UNIVERSE

Over 200 scientists and other interested persons attended a symposium titled "Prospects for Life in the Universe: The Ultimate Limits to Growth" on 15 February 1978 in Washington, D.C. Organised by William Gale of Bell Telephone Laboratories, the symposium was part of the annual meeting of the American Association for the Advancement of Science.

Jesco von Puttkamer of NASA's Office of Space Flight discussed "The Industrialization of Space: NASA Plans for the Next 25 Years." He presented a chart showing the steps which humanity must pass through to open a wide range of choices in space after the year 2000, and stressed that all models must pass through the "needle's eye" of a space station. Industrialization is a realistic approach which can lead to permanent, practical, commercial uses of space that will create new values, new jobs, and a better quality of life for Mankind. It can lead to self-sufficient settlements in space and, by being non-elitist, can begin the true humanisation of space. Von Puttkamer warned, however, that there was little public support for large-scale enterprises in space (the funding for NASA's advanced planning activities has been cut in half for fiscal year 1979).

Brian O'Leary, an ex-astronaut now with the Princeton University Physics Department, discussed how the mining and processing of extraterrestrial materials will allow the building of large structures in space, such as satellite solar power stations, that can help relieve the limits to growth on Earth. O'Leary, who is a close associate of space colony proponent Gerard O'Neill, argued that the transition to a steady-state, no-growth world may be socially impossible. He stressed the mining of the asteroids, an area in which he has published research work. O'Leary showed artist's visualisations of how small asteroids might be retrieved by a mass-driver vehicle, which would tow them to processing facilities in cis-Lunar space.

William Gale described how human colonization might extend throughout the Solar System, to other stars, and ultimately throughout the Galaxy. The complete utilisation of the physical resources of the Solar System could support a sextillion humans at a high standard of living, and could be achieved in 3,000 years. At average expansion velocities of 0.01 to 0.1 c, the Galaxy could be colonized in 1 to 10 million years. Expansion to other galaxies might follow, presumably at higher average velocities. Other intelligent species presumably expand too, and may attempt to reshape their environments on a large scale, perhaps into Dyson spheres. If the intergalactic travel speed is appreciably less than the speed of light, infrared galaxy clusters should be visible. Since we see no evidence of such macroengineering, intelligent life in the Universe must be extremely rare – no more than one civilisation in every 10,000 galaxies. We can thus anticipate millions of years of growth for the human species.

In a talk on "Anthropophysics," Greg Edwards of the National Science Foundation discussed the importance of societal expectations to future development. He concluded that Mankind eventually would be forced to move outward; the only question is when. He added that the only limits to growth are hope, heart, and chutzpah.

In a brilliant and witty humorous paper, Freeman Dyson of the Institute for Advanced Study at Princeton argued that, in an open cosmology, the complete loss of the structured energy of the Universe would require almost inconceivably long periods of time (in one example, ten to the 10th to the 76th years). A civilization which husbands its resources conservatively can continue to process information forever using a finite store of energy.

Discussant Carl Sagan of Cornell University emphasised the humanistic aspects of our relationship with the Universe. He stressed the importance of solving current world problems to assure the survival of our civilization, and commented that space colonies could not solve the Earth's population problem. Discussant Michael Michaud of the U.S. Department of State argued that what was really being proposed was a long-term goal for humanity; getting the human expansion under way would require a massive educational effort and changes in human values and politics. Michaud expressed the hope that the intelligences of the Universe would try to work together for the survival of intelligence itself, and not regard survival as a zero-sum game.

Other AAAS sessions of interest to JIBS Red Cover Issue readers included "New Tools for Viewing the Universe"; "Microengineering Projects – The Infrastructure of Tomorrow"; "Gravitational Physics – A New Window on the Universe"; "Progress in X-ray Astronomy"; "Humans in the Cosmos"; and "Solar System Exploration – Should it be a National Commitment?" One session, titled "The Search for Extraterrestrial Intelligences: Priority to Pandora’s Box?", was conducted by linking the AAAS meeting in Washington with the NASA Ames Research Center in California by satellite television, through the Communications Technology Satellite. A soft-cover book containing the abstracts of all papers presented at the AAAS annual meeting is available for U.S. $8.00 from the American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W., Washington D. C. 20005, U.S.A.

M. A. G. MICHAUD