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MAXIMUM TERMINAL VELOCITY OF RELATIVISTIC ROCKET*

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Abstract—The maximum terminal velocity problem of the classical propulsion is extended to a relativistic rocket assumed broken down into active mass, inert mass and gross payload. A fraction of the active mass is converted into energy shared between inert mass and active mass residual. Significant effects are considered. State and co-state equations are carried out to find the exhaust speed optimal profile.

A first major result consists of a critical value of inert mass. Beyond it both true and effective jet speeds increase with time. Below it the true jet speed profile is reversed. At criticality, the best control consists of both velocities constant in time.

A second meaningful result is represented by an interval of inert mass outside which no optimal control exists. Numerical results are discussed with particular emphasis to current concepts of antimatter propulsion.

NOTATION

- α alpha
- β beta
- γ gamma
- ε epsilon
- σ sigma
- λ small lambda
- Λ capital lambda
- τ tau
- Δ capital delta
- χ chi
- ∂ partial differentiation
- ln natural logarithm
- $\rightarrow$ arrow
- $\Rightarrow$ double arrow

1. INTRODUCTION

The problem of maximizing the terminal velocity of a space vehicle has been dealt with extensively in literature. A particular attention has been devoted to both electric and nuclear propulsions for interplanetary and out-of-solar-system missions. Here we cope with this problem in a general way. Our approach is independent of the particular spacecraft one could consider, provided only that the power source for propulsion is on-board. Therefore we will not consider vehicles receiving momentum and/or energy from outside. These last types of propulsion have been included in [1].

The relativistic formulation of this problem is considered here. This is the driving purpose of such a paper. In addition, a second goal consists of introducing concepts relevant to high energy density source rockets.

Basic concepts and equations of special relativity can be found in a number of excellent textbooks, e.g. [2].

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2. STATEMENT OF THE PROBLEM

Any spacecraft (S/C) endowed with a pure rocket propulsion system can be broken down into three main systems in terms of mass: the active mass from which the propulsive energy is extracted, the inert mass to which this energy is generally transferred and then exhausted and the gross payload. A key point is to establish the mass-energy utilization history in the S/C. In [3] a 14-parameter model is considered. Those parameters account for effects such as mass jettisoning, leakage, nonpropulsive energy, nozzle spreading and so forth. That model contains specifications relevant to an antimatter propulsion system concept.

In the present context we simplify that model by retaining only three key parameters. They are redefined here.

Figure 1 shows the mass-energy distribution for thrusting. It is valid for both continuous-mode and pulsed-mode propulsions. The continuous-mode is referenced here for greater clarity.

The fraction ε of the active mass M_a is converted into energy by some exothermic reaction. This fraction ε encompasses the rest and kinetic energy of all products but the interaction-negligible particles. The inert mass M_i receives a fraction of this energy and then it is ejected. Sometimes the active and inert mass physically coincide. Also, the inert mass could be absent.

A special situation would consist of the (optional) capability of a propulsion system to reconvert a fraction, say σ, of the energy M_a into nonzero rest-mass particles in order to make a (generally additional) controllable ejection beam. The σ value can be generally composed of two terms of different physical origin. We denote them by σ_{sp} and σ_{ind}, where "sp" and "ind" stand for spontaneous and induced, respectively. The former term represents the fractional rest mass of particles directly produced. In fact, a complex reaction generally yields massive products. On the other hand, some reaction products would be completely lost if not reconverted into charged

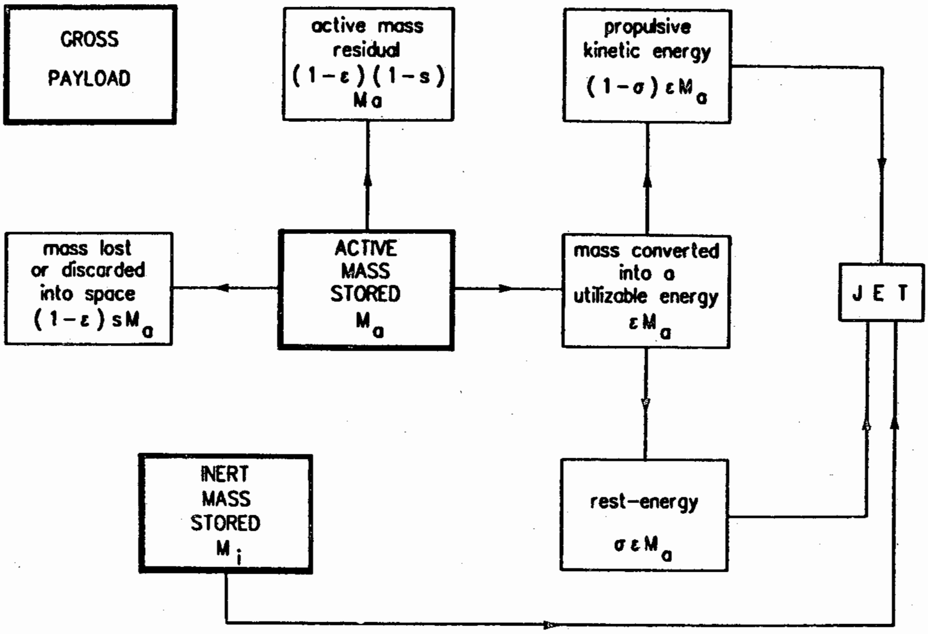


Fig. 1. Spacecraft mass breakdown and energy utilization history.

massive particles (e.g. high-energy gamma rays transformed into electron-positron pairs). The fraction of the total energy of these particles going to rest mass upon transformation is just the term σ_{ind}. Therefore, the overall ejection beam can contain "sub-beams" originated from the inert and active mass.

In high-energy exothermic reactions massive particles can be the final products of some chain of secondary reactions. In intermediate phases a fraction of the total energy may be quite lost because some particles have very low interaction cross sections (e.g. neutrinos) or they are hard directable. The fraction ε is an effective value depending upon the energy distribution in the chain. These considerations introduce the "jettisoning" factor, say s, into the current model; s represents the fraction of the mass $(1 - \varepsilon)M_a$ lost into space at zero total momentum with respect to the S/C frame.

In order to write the mass-energy conservation laws we refer to an inertial heliocentric frame (HF) and the set of the instantaneously inertial ship frames (SF) as usually defined in special relativity. We have in differential form (the speed of light has been set equal to one) [1]:

$$\begin{aligned} dE_s &= M\,V\,\gamma_v^3\,dV + \gamma_v\,dM, \\ dE_p &= (1 - U\,V)\,\gamma_u\,\gamma_v\,dM_p, \\ dE_j &= (1 - \varepsilon)\,s\,\gamma_v\,dM_a, \end{aligned} \tag{1}$$

where $dM_p = dM_i + \sigma\,\varepsilon\,dM_a$. The symbol M denotes the ship (rest) mass, V its velocity, U the true jet speed. The subscripts s, p, j stand for ship, propellant and jettisoning, respectively. The Lorentz factor is denoted by $\gamma_{(\cdot)}$. Equations (1) are valid in HF. In SF the mass-energy conservation is to be read as follows:

$$dM = -(1 - \varepsilon)\,s\,dM_a - \varepsilon\,dM_a - dM_i, \tag{2}$$

$$(1 - \sigma)\,\varepsilon\,dM_a = (\gamma_u - 1)\,dM_p. \tag{3}$$

Introducing the variable $\chi = dM_i/dM_a$, the kinetic energy eqn (3) allows us to immediately obtain:

$$U^2 = 1 - [(\chi + \sigma\varepsilon)/(\chi + \varepsilon)]^2. \tag{4}$$

From the basic equation:

$$dE_s + dE_p + dE_j = 0, \tag{5}$$

we can obtain the rocket motion equation in differential form:

$$\gamma_v^2\,dV = -(U\,dM/M)\,(\chi + \varepsilon)/[\chi + \varepsilon + (1 - \varepsilon)s] \triangleq -U_e\,dM/M. \tag{6}$$

The second fraction in the right-hand side of eqn (6) can be referred to as the utilization efficiency relative to the total mass released outside the ship. It equals one when $\varepsilon = 1$ or $s = 0$, being in general time-dependent as well as U. Note that whereas U is independent of the value s, the above efficiency does not depend upon σ. Only the effective speed, U_e, depends on both.

We maximize

$$J = \int_0^{\tau_f} \gamma_v^2\,\dot V\,d\tau = \tanh^{-1}(V)\Big|_{V_0}^{V_f}, \tag{7}$$

subject to the integral constraint:

$$\int_0^{\tau_f} \chi\,\dot M_a\,d\tau = M_i, \tag{8}$$

and the differential constraint, from eqn (2):

$$\dot M = -[(1 - \varepsilon)s + (\chi + \varepsilon)]\,\dot M_a. \tag{9}$$

The proper propulsion time τ_f is fixed. In eqns (7) through (9) the dot represents differentiation with respect to the proper time τ.

If we assume $\dot{M}_a$ = const. = M_a/τ_f and assigned, the only degree of freedom of the system is χ. However, for analytical purposes it is easier to take as control variable γ_u which is related to χ through

$$(\gamma_u - 1)(\chi + \sigma\varepsilon) = (1 - \sigma)\varepsilon, \quad (10)$$

as easily obtainable from eqn (4). Thus the state equation (9) can be recast into the form:

$$\dot{M} = -\dot{M}_a[(1 - \varepsilon)s + (1 - \sigma)\,\varepsilon\,\gamma_u/(\gamma_u - 1)]. \quad (11)$$

Then, by inserting eqn (4) into the definition of γ_u the Hamiltonian H of the current problem can be expressed as follows:

$$H/\dot{M}_a = (1 - \sigma)\varepsilon[(\gamma_u + 1)/(\gamma_u - 1)]^{1/2}/M + \lambda[(1 - \sigma)\varepsilon\,\gamma_u/(\gamma_u - 1) - \varepsilon] - \Lambda[(1 - \varepsilon)s + (1 - \sigma)\varepsilon\,\gamma_u/(\gamma_u - 1)], \quad (12)$$

where the expressions in γ_u are easily obtainable combining eqn (4) and the definition of γ_u. The multipliers λ and Λ are time-independent and time-varying respectively. H is different from zero because τ_f is assigned.

3. OPTIMAL EQUATIONS

No (sub-C) limit on the exhaust velocity is set. Because H contains neither a linear control nor any limit on it, the problem is nonsingular and the optimal program is given by $\partial H/\partial\gamma_u = 0$. This control equation can be arranged into the form:

$$\Lambda - \lambda = [(\gamma_u - 1)/(\gamma_u + 1)]^{1/2}/M. \quad (13)$$

After differentiating eqn (13) with respect to time, noting $\dot{\lambda} = 0$ identically and making explicit the mass adjoint equation $\dot{\Lambda} = -\partial H/\partial M$, we arrive at the following differential equations:

$$\dot{\Lambda} = [(1 - \sigma)\varepsilon\,\dot{M}_a/M^2] \times [(\gamma_u + 1)/(\gamma_u - 1)]^{1/2}, \quad (14)$$

$$\dot{\gamma}_u = (\dot{M}_a/M)\,[(1 - \sigma)\varepsilon \times (\gamma_u + 1) - (1 - \varepsilon)s(\gamma_u^2 - 1)]. \quad (15)$$

Finally, the proper acceleration is obtained from eqn (6) as function of the control as follows:

$$\dot{V} = \gamma_v^{-2}[(1 - \sigma)\varepsilon\,\dot{M}_a/M]\,[(\gamma_u + 1)/(\gamma_u - 1)]^{1/2}. \quad (16)$$

Equations (11), (14)–(16) represent the optimal system which provides the best control to maximize the final velocity of a relativistic ship.

Some remarks are in order. First, only eqns (11) and (15) are coupled. This means that, if the initial value of γ_u were known, one could easily carry out the optimal profile. However, it is not possible to express $\gamma_u(0)$ analytically as function of known quantities such as M_0 and M_f. As a matter of fact, H does not contain time explicitly; then, inserting Λ from eqn (13) into the H expression (12) and considering $H = H_0$, one obtains:

$$(a + \beta - \beta X)/[M\,(X^2 - 1)^{1/2}] = (a + \beta - \beta X_0)/[M_0(X_0^2 - 1)^{1/2}], \quad (17)$$

where we have set $X = \gamma_u$, $a = (1 - \sigma)\varepsilon$, $\beta = (1 - \varepsilon)s$.

In principle, if X_0 were known, eqn (17) would give the optimal X as function of M. Equation (11) would be the only one to be integrated. However, X_0 cannot be generally obtained in closed form as function of ε, σ and s, values characteristic of a given propulsion system. Therefore, starting from a guess X_0, the basic system to be integrated is composed of eqns (11) and (15) plus their respective *variant trajectories*. Successive differential corrections bring X_0 to converge (note that Λ_0 is arbitrary).

The integral constraint (8) could be chosen equal to zero, that is no inert mass is considered. From eqn (10) one immediately concludes that the true jet speed is constant (and equal to the speed of light if σ is zero). By a physical analogy, one could then ask whether an optimal time-independent profile may occur under more general conditions. For eqn (15) admits the solution γ_u = const only if the term in brackets vanishes. One value of γ_u is acceptable:

$$\gamma_u^* - 1 = a/\beta. \quad (18)$$

From a pure mathematical viewpoint, eqn (18) can be simply obtained by equating the right-hand side of eqn (17) to zero. Physically, equation (18) implies that there is a critical value of M_i, say M_{ic}, which ultimately determines the trend of the optimal profile.

We get from eqn (10):

$$M_{ic} = M_a[s - \varepsilon(s + \sigma)]. \quad (19)$$

One realizes that for an actual $M_i < M_{ic}$ the profile of U is reversed, namely, the initial values of U are greater than its final ones. In contrast, U_e never decreases, as expected. If $M_{ic} < 0$, we have a classical profile (i.e. time-increasing) for any actual M_i.

What is the physical cause of a critical χ, say χ^*, which equals $[s - \varepsilon(s + \sigma)]$? Recalling eqn (10) let us note that the fractional (rest) mass accelerated up to the speed U is $\chi + \sigma\varepsilon$. This value, in critical conditions, results simply in:

$$\chi^* + \sigma\varepsilon = s(1 - \varepsilon). \quad (20)$$

The right-hand side of eqn (20) is the total fractional energy lost into space. Therefore, if one adds to the active mass flow ($\sigma\varepsilon$) an amount of extra (inert) mass such that the overall loss of energy into space is exactly compensated, the optimal exhaust speed is a constant profile. One can recognize that only for a high-ε reaction the value of s may be sufficiently high to entail χ^* significantly positive. In low-ε environments any actual

s approaches zero and χ^* is nonpositive. This is essentially why "classical engines" such as electric devices exhibit a unique optimal trend of specific impulse for max-ΔV.

Let us examine the optimal equations at criticality. Here the solution is in closed form. From equations (10), (11), (15), (16) one can carry out:

$$U = [a(a + 2\beta)]^{1/2}/(a + \beta), \quad (21)$$

$$M = M_0 - \dot{M}_a(a + 2\beta)\tau, \quad (22)$$

$$\Delta \tanh^{-1} V = (1 + 2\beta/a)^{-1/2} \ln (R), \quad (23)$$

where R denotes the propulsion mass ratio. In general, R is expressed by $M_o/[M_o - M_i - \varepsilon M_a - s(1 - \varepsilon)M_a]$. Equations (21) and (23) show that both U and U_e are time-constant. A special case is $M_i = M_{ic} = 0$ (then $s = \varepsilon\sigma/(1 - \varepsilon)$). Equation (23) results in:

$$\Delta \tanh^{-1} V = [(1 - \sigma)/(1 + \sigma)]^{1/2} \ln (R). \quad (24)$$

The (rest) mass-flow rate can be expressed as follows:

$$\dot{M}_p = \sigma\varepsilon \dot{M}_a, \quad (25)$$

whereas the on-board acceleration is given by:

$$a_s = (\dot{M}_a/M)\varepsilon(1 - \sigma^2)^{1/2}. \quad (26)$$

In these environments the true jet speed results in:

$$U = (1 - \sigma^2)^{1/2}. \quad (27)$$

Where σ equals zero (then $s = 0$), one would again find the ideal photon rocket.

4. POTENTIAL APPLICATIONS

Extending the max-terminal velocity problem to rockets for which the effective energy conversion yield is high, the classical solution generalizes into three possible regions of operation. In the overcritical region the relativistic control repeats the classical one. In the undercritical region the true jet speed program is reversed; in contrast, the effective jet speed increases as burning progresses. Finally, in the critical regime both optimal controls are time-constant. These are strict results.

The critical environment is driven by eqn (19). It has no practical importance in classical propulsion (electromagnetic and nuclear systems included) because the amounts of propellant involved in the corresponding missions envisaged are considerably greater than the related critical masses. Thus, the question arises what might be a possible area of future application. Table 1 displays the values of the energy yield for the main exothermic processes in nature.

In the last two decades, several concepts for fast interplanetary and interstellar missions to nearby stars have been developed. Recently, investigators in the USA and Europe have performed studies about antimatter propulsion[4–10]. This propulsion concept is highly attractive essentially for the very high energy density that nucleon–antinucleon annihilation can offer. However, the involved problems are the most challenging in theory and technology.

Table 1. Major energy sources

TYPE	MAX USEFUL ENERGY (MJ/Kg)	EFFECTIVE YIELD
CHEMICAL (LH_2/LOX)	15	1.7 E − 10
FREE RADICAL ($H + H \rightarrow H_2$)	220	2.4 E − 09
METASTABLE ATOM (He^3)	480	5.3 E − 09
NUCLEAR FISSION	8.2 E + 07	9.1 E − 04
NUCLEAR FUSION	3.9 E + 08	4.3 E − 03
PROTON-ANTIPROTON ANNIHILATION	9.0 E + 10*	0.55 − 1

*Per kilogram of matter–antimatter combined.

Figure 2 shows significant (mean) values of a proton-antiproton annihilation-at-rest reaction. A low-energy proton-antiproton annihilation largely produces a number of neutral and charged pions. The neutral pion decays into two high-energy gamma rays; the charged pions decay into muons and neutrinos; the muons, in turn, give electrons, positrons and neutrinos as final products. Because of such temporal sequence of events, the energy efficiency of an antimatter engine would be strongly dependent upon the type of particles which ultimately furnish the propulsive energy, especially if the annihilation products were to be exhausted with no additional component of ordinary matter[7–10]. The energy reconversion concept could improve this efficiency[10].

5. NUMERICAL RESULTS

A computer code has been implemented to solve for the general optimal equations. Besides the usual printout, a three-dimensional graphic representation output option provides the analyst for single-graph and multiple-graph displays. If a single case is selected, the scaling factors are explicitly written. When several cases are to be compared, the critical mass is reported. Title and label explanation can be found below each graph frame.

The multiple-case figures presented here are characterized by the projections of the optimal behaviours on the coordinate planes. The indicated scales are normalized by the following quantities (the optimal solution can be set in a dimensionless form): the prefixed thrusting time, the initial ship mass, the maximum value of the inert mass on active mass ratio. Velocities are expressed in speed-of-light unit. Path is expressed in lightyear.

Figure 3 shows the optimal profile concerning a spacecraft which, ideally, exploits the whole energy released in the nuclear fusion. The fractional kinetic energy available amounts to 0.004 [Because the whole fuel mass is ejected, we set $\varepsilon = 1$ and $\sigma = 0.996$ so that $a = (1 - \sigma)\varepsilon = 0.004$. In addition, in the ideal case

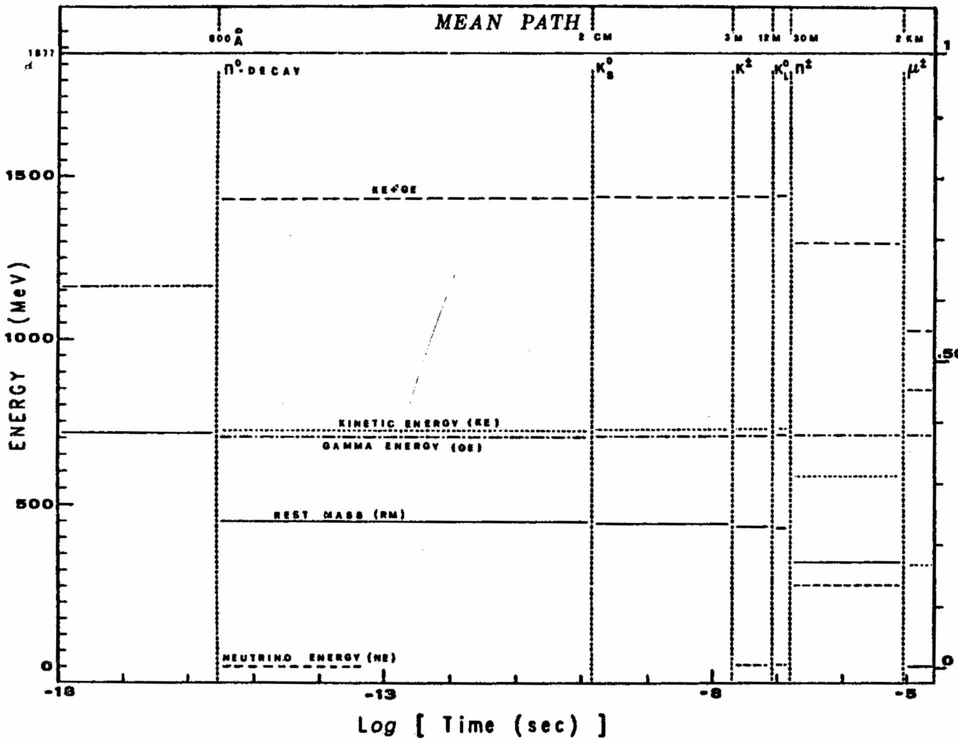


Fig. 2. Energy sharing following the average reaction of proton-antiproton annihilation at-rest. The symbol π denotes a pion, $K^{\pm}$ does charged kaons (K-mesons), K_S^0 and K_L^0 indicate the two states, of short and long life respectively, by which a neutral K-meson can be described. Pions and kaons represent the nonresonant primary product of a nucleon-antinucleon annihilation.

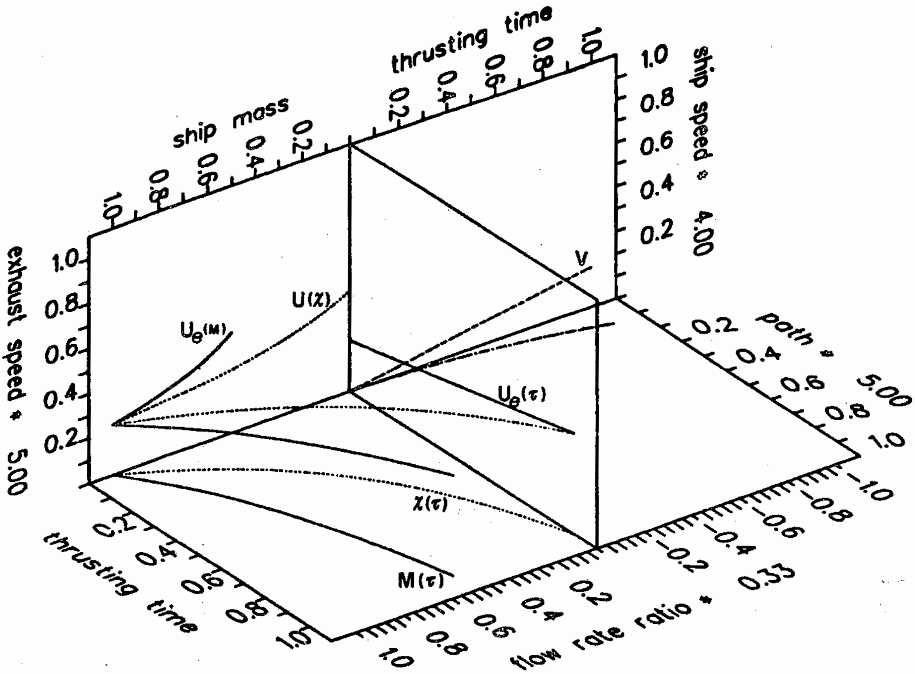
$s = 0$. More generally, the case of an incomplete ejection of the fusion mass can be dealt with by decreasing ε and increasing s appropriately.] The critical mass equals $-\sigma\ M_a$. The optimal control is therefore of classical type. In order to have an increasing jet speed we must add inert mass. A modulated χ is then allowed, as shown in Fig. 3. An interesting effect has been emphasized here in the behaviour of χ and, in turn, in the U's. The terminal χ is very low. If we augment M_i beyond a certain value, say M_{iu}, the mass-flow rate ratio crosses zero and takes on negative values. This has no physical meaning. The upper limit on M_i is to be ascribed to the fact that $\dot{M}_p \geqslant \sigma\varepsilon\dot{M}_a$ in this type of engine. In contrast, in a nuclear electric propulsion (NEP) system (where the active mass *does not participate* in the jet) the propellant flow rate can in principle be made arbitrarily low, as it is known. A fission NEP S/C is characterized by the following model values: $\varepsilon = 0.001$, $\sigma = 0$, $s = 0$. The M_{ic} value is identically zero.

The energy density of a power source is chiefly determined by the factor a relevant to the exothermic reactions the power plant utilizes. A very high a, the highest one known at present, pertains to the annihilation process. Figures 4 through 7 present the optimal behaviours of envisaged matter-antimatter powered spaceships. The first three diagrammes refer to $\sigma_{ind} = 0$. A partial reconversion of the gamma energy into relativistic electron-positron pairs is considered in the fourth one. The basic energy source is the proton-antiproton annihilation at rest. The values of ε and σ are reported from[9].

Figure 4 regards a spaceship endowed with a propulsion system capable of utilizing the energy of the charged pions. The neutral pions have already decayed in gammas supposed quite lost in this example. The quantities $\varepsilon = 0.618$ and $\varepsilon_m = 0.23$ represent the total energy and the rest-energy, respectively, of the charged pions in units of annihilation energy (the måss of two protons, namely, about 1877 MeV). The propulsion kinetic energy available is $(1 - \sigma)\varepsilon = 0.388$ (about 100 times the best fusion engine). The value of s equals one. The value of the active mass has been chosen rather high (0.5) for a greater clarity of graphic presentation. The ship is a relativistic spacecraft. The inert mass has been varied to show the regions of optimality and other effects.

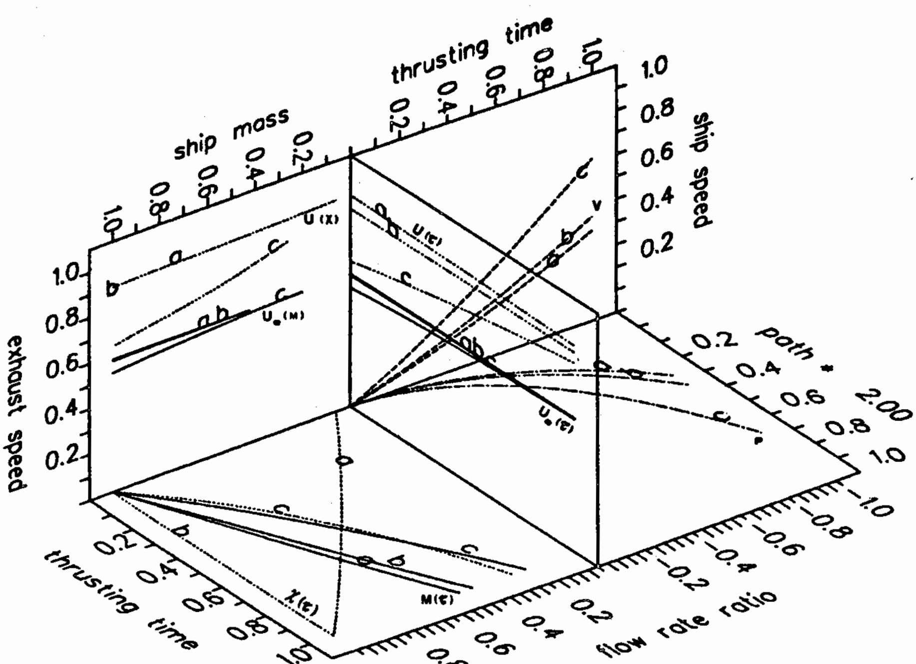
In Fig. 4 the undercritical, critical and overcritical behaviours are shown. The effective jet speed is practically insensitive to the inert mass value unless it is comparable with the active mass. Particularly interesting is the convexity of $\chi(\tau)$. The sign of such convexity is constant with respect to both time and inert mass.

Figure 5 displays the optimal trend in the energy region where the charged pions are decayed, but the muons are not. The available propulsive kinetic energy is $a = 0.312$; however, the total energy drops significantly on account of neutrinos produced by pion decay. This means a strong increase of the critical mass. As a consequence, case (a) exhibits a negative flow rate ratio. These considerations can be repeated for the behaviour (a) in Fig. 6. There the optimal trends are relevant to an antimatter engine which exploits the energy of the electrons and positrons



---- (time,true jet,flows) — (time,eff. jet,mass) --- (time,ship vel.) --- (time,path)
Ma=0.250 Mi=0.250 s=0.000 (ε σ τ) ≡ 1.0000 0.996 1.000

Fig. 3. Ideal fusion engine spacecraft. A modulation of the true jet speed is obtained by adding inert mass to the basic flow of active mass. This option may result of practical importance if some constraint were to limit the active mass on-board for a certain mission.



---- (time,true jet,flows) — (time,eff. jet,mass) --- (time,ship vel.) --- (time,path)
Critical Mi = 0.076 Ms=0.500 s=1.000 (ε σ τ) ≡ 0.6180 0.372 1.000
Mi varied [a] Mi=0.019 [b] Mi=0.076 [c] Mi=0.300

Fig. 4. Antimatter propulsion spacecraft utilizing the energy of the charged pions. The gamma and neutrino energy is lost.

from the muon decay. Figures 5–6 (and other similar results not reported in this paper) show the existence of a lower limit of M_i, say M_{il}, at which the initial χ is exactly zero. M_{il}, in high-ε environments, is the counterpart of M_{iu} in low-ε environments. Such values are solutions of the following equation:

$$\Gamma_0 \Gamma_f (M_i + \sigma\varepsilon M_a) = a\,(\gamma_0\,\Gamma_f\,M_0 - \gamma_f\,\Gamma_0\,M_f), \qquad (28)$$

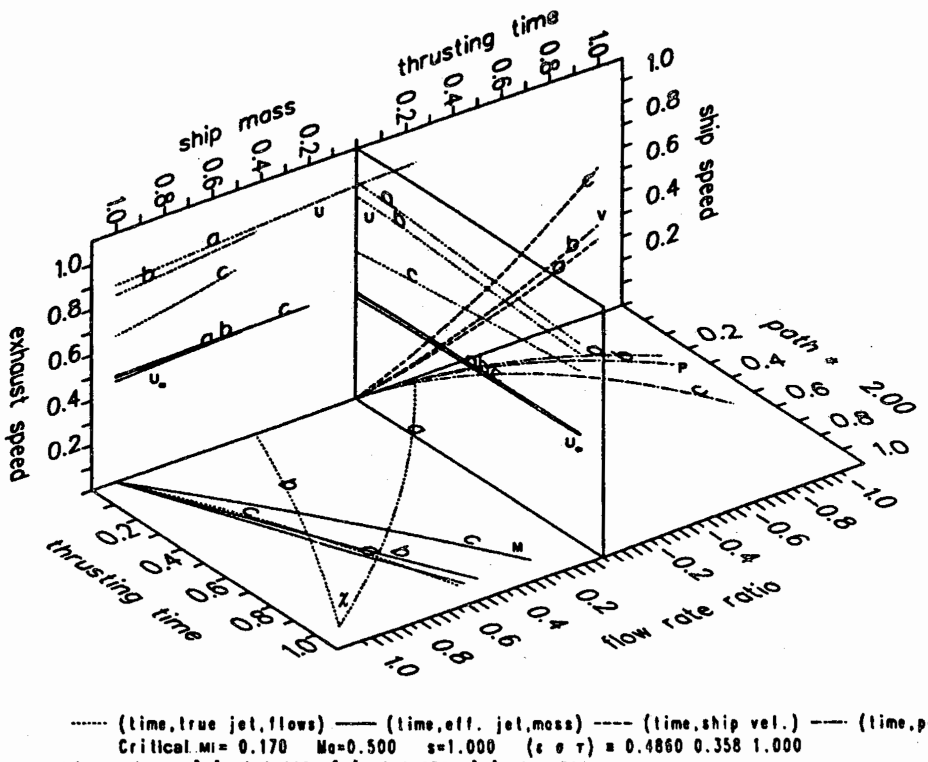
where we have set $\Gamma = (a + \beta - \beta\gamma)$.

Equation (28) has been obtained by combining eqns (10), (15), (17) and integrating by variable separation. M_{il} corresponds to $\gamma_0 = 1/\sigma$ whereas M_{iu} does to $\gamma_f = 1/\sigma$. The other endpoints, respectively, can be obtained from eqn (17). Equation (28) is to be solved numerically. For the set of values of Fig. 5 we get $M_{il} = 0.033\,M_0$.

The interval $[M_{il}, M_{iu}]$ represents the admissible band for the inert mass to have a physically acceptable optimal control. The trend of such control is driven by the sign of the difference $M_i - M_{ic}$. Let us notice that for a NEP spacecraft $M_{il} \rightarrow 0$ and $M_{iu} \rightarrow 1$, that is the admissible range is ideally the whole vehicle mass. This would explain why this "band effect" has not been noted in classical propulsion.

Figure 7 represents an envisaged matter-antimatter S/C where 50% of the gamma energy is utilised for propulsion (this most probably is a limit in effecting the reconversion process. Realistic values could range from 20% to 30%[10]). Because of the gamma conversion, assumed to be accomplished by means of light particles (i.e. electrons and positrons), the critical mass decreases down to a negative value. The max-velocity is then achieved by means of a classical-type control.

Finally, the current formulation of the max-ΔV problem can be further generalized by allowing the sum, say m, of the active and inert mass to be fixed and leaving M_a free. The control consists of $\dot{M}_a$ and $\chi(\tau)$. Although unknown, $\dot{M}_a$ is assumed to be constant during the burning. The problem is partially singular because the Hamiltonian is linear in $\dot{M}_a$ and nonlinear in χ (or γ_u). Instead of dealing again with the analytical problem, we have examined a number of cases by running the previous computer code several times. Focusing our attention on the antimatter propulsion once again, we have obtained the graphic output shown in Fig. 8. The value $m = M_a + M_i = 0.5$ has been selected. The starred marks in Fig. 8 represent parabolic least-square fits giving an approximate location of the maxima, namely, max[max-ΔV] values. A number of features can be drawn. The absolute maxima fall in the overcritical region. They are very wide and broaden as the propulsion energy increases. As a consequence, it is not useful to augment the active mass beyond a certain level. This would limit to a certain extent the antimatter amount and its complex management on-board. This sort of saturation effect takes place because at low M_i the jet speed variation throughout the thrusting is lower than at high M_i.



······ (time,true jet,flows) —— (time,eff. jet,mass) ---- (time,ship vel.) —·— (time,path)
Critical Mi= 0.170 Ma=0.500 s=1.000 (ε σ τ) = 0.4860 0.358 1.000
Mi varied [a] Mi=0.019 [b] Mi=0.076 [c] Mi=0.300

Fig. 5. Antimatter propulsion spacecraft utilizing the energy of the muons. The gamma and neutrino energy is lost.

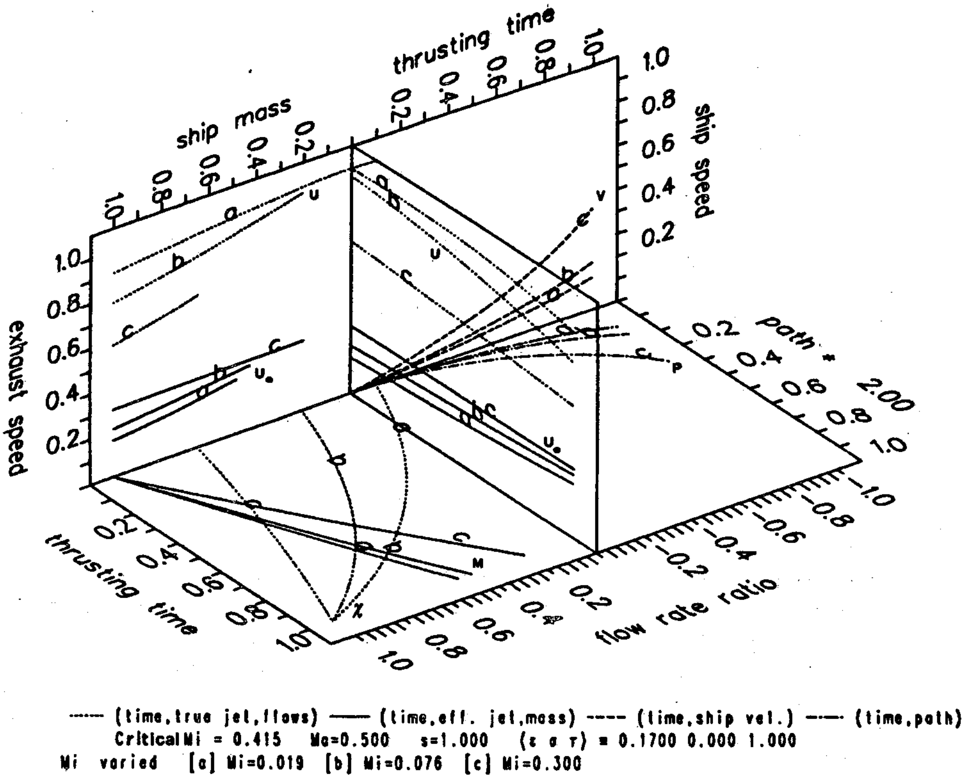


Fig. 6. Antimatter propulsion spacecraft utilizing the energy of the electrons and positrons. The gamma and neutrino energy is lost.

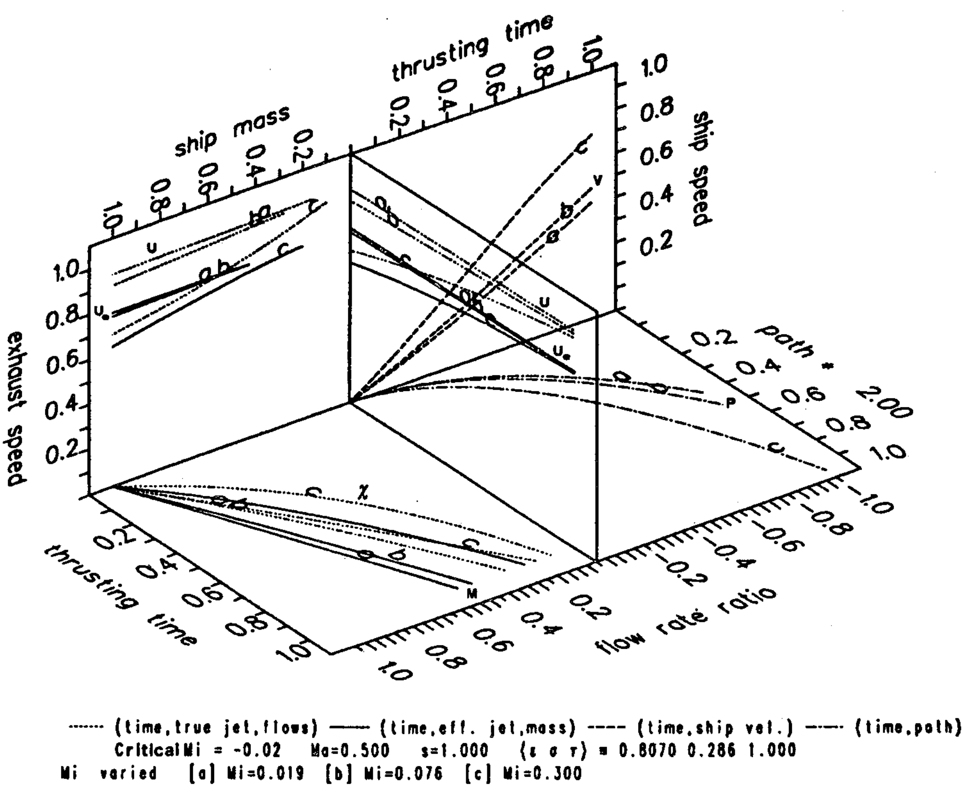


Fig. 7. Antimatter propulsion spacecraft utilizing the energy of the charged pions and one-half of the gamma energy amount (compare with Fig. 4).

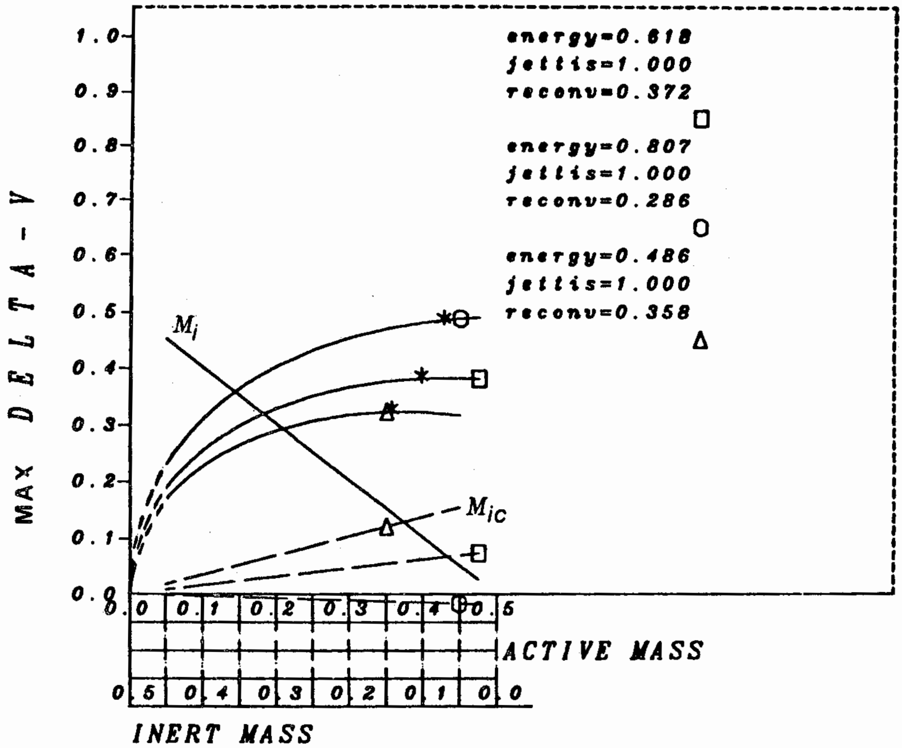


Fig. 8. Antimatter propulsion spacecraft: the active mass is allowed to vary, but the sum of inert and active mass is fixed.

6. CONCLUSIONS

A general model of pure rocket has been presented. It has been applied to the field-free maximum terminal velocity problem extended to relativistic environments. Few key parameters have been used to describe the model and the energy utilization history. Any exothermic reaction can be accounted for in the current description, although an explicit reference is made to high-energy yield reactions.

After having formulated the optimal problem, numerical results have been carried out, largely oriented to potential antimatter-powered rockets the fundamentals of which are being developed in the USA and Europe.

From a general viewpoint the current extended formulation of the maximum final ship velocity problem has displayed a number of new aspects not "detectable" in classical propulsion environments. The major aspect is perhaps the existence of a critical value of the inert component of the propellant beyond which both true and effective jet speeds are time-increasing at optimality. Below such a value, the trend of the true exhaust speed is reversed. At the critical point the optimal jet speed is strictly constant. In addition, there exists an interval or band of inert mass outside which no optimal control for maximizing the delta-V is physically possible.

Such features do not matter in classical propulsion systems for which the energy yield is generally low and/or the active mass does not participate in the exhausting beam.

The classical rocket optimal jet profile has been found again from the general equations under a few simple conditions (see Appendix).

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APPENDIX: OPTIMAL ANALYTICAL SOLUTION

Here we specialize the optimal equations of Section 3 to the case $s = 0$, $\sigma = 0$ and $\varepsilon \ll 1$. As a consequence $a = \varepsilon$, ε so representing the fractional propulsive kinetic energy. The s value entails that the mass $1 - \varepsilon$ is onboard. Therefore, a nonzero M_i must be given the energy a. In other words, we are dealing with a power source the output of which is transferred to an inert mass, unique component of the jet. If we pick out $\varepsilon = 0.001$, one can immediately identify this case with an NEP system. The optimal delta-V is then obtained by Stuhlinger's famous profile of the exhaust speed[11]. We find again such program by starting from our general equations.

For eqns (11) and (15) result in:

$$U \dot{U} = 2 \varepsilon \dot{M}_a / M, \tag{A-1}$$

$$\dot{M} = -2 \varepsilon \dot{M}_a / U^2. \tag{A-2}$$

By a direct substitution of $\varepsilon \dot{M}_a$ from (A-2) into (A-1) we have:

$$\dot{U}/U = -\dot{M}/M \quad \Rightarrow U = U_0 M_0 / M, \tag{A-3}$$

which is the well-known profile. In addition, eqn (10) gives:

$$U = (2\varepsilon/\chi)^{1/2} \quad \Rightarrow \chi = \chi_0 (M/M_0)^2. \tag{A-4}$$

By substitution of the U program into (A-2) we get:

$$U_0^2 = 2\varepsilon\, M_a (M_f/M_0)/(M_0 - M_f), \tag{A-5}$$

or, equivalently,

$$\chi_0 = (R - 1)\, M_0/M_a, \tag{A-6}$$

where $R = M_0/M_f$ is the propulsion mass ratio, which is known: $M_f = M_0 - M_i - \varepsilon\, M_a \simeq M_0 - M_i$. The explicit temporal profile of the specific impulse is then carried out by a simple quadrature. It results in:

$$U(t) = \overline{U}\, [R^{-1/2} + (R^{+1/2} - R^{-1/2})\, t/t_b], \tag{A-7}$$

where $\overline{U} = (2\varepsilon \dot{M}_a t_b/(M_o - M_f))^{1/2}$ and t_b is the thrusting time. Note that $\varepsilon \dot{M}_a$ is just the effective power released by the nuclear reactor. The time profile of χ can be obtained from eqns (A-7) and (A-4). In the current context, as well as in Stuhlinger's, the propellant is assumed to be fully accelerated. This entails $U = U_e$. The present profile can be generalized in a wider NEP context such as the subject dealt with in [12].

Add-on to Appendix-A

Although some relationships are independent of the particular units system one may choose, however in general one should remind:

$c \equiv 1 \qquad 1\,year \equiv 1$

and replace:

$\dot{M}_a \;\rightarrow\; \dot{M}_a c^2$ and $U_x \rightarrow U_x c \quad \left(x \in \{0, f\}\text{.OR. } \varnothing\right)$

$t \;\rightarrow\; t \cdot year[\text{SI}]$

in order to restore the equations in the paper to SI

$$R \equiv \frac{M_0}{M_f} = 1 + \frac{2\varepsilon \dot{M}_a t_b}{M_0 U_0^2} \;, \qquad U_0{}^2 = \frac{2\varepsilon \dot{M}_a t_b}{M_0 - M_f} \frac{M_f}{M_0}$$

$$U(t) = U_0 + \frac{2\varepsilon \dot{M}_a}{U_0 M_0} t \;, \qquad U_f = U_0 M_0 / M_f = U_0 R$$

$$M(t) = \frac{M_0}{1 + \dfrac{2\varepsilon \dot{M}_a}{U_0{}^2 M_0} t}$$

$$\langle U \rangle = (R+1)\sqrt{\frac{\varepsilon \dot{M}_a t_b}{2M_0 (R-1)}} \;, \qquad \langle V \rangle = (R-1)\sqrt{\frac{\varepsilon \dot{M}_a t_b}{2M_0 (R-1)}}$$

$$\frac{\langle V \rangle}{\langle U \rangle} = \frac{R-1}{R+1}$$

where $\langle \;\; \rangle$ denotes the average over the burning time $[0, t_b]$.